PHY7400. Homework 2

This homework assignment is due on **October 3**.

Suggested reading:

Eugen Merzbacher, *Quantum Mechanics*, Chapters 4-5.

Problem 1: Fun with unitary operators (10 pt).

We spent much time discussing the properties of various operators used in quantum mechanics. In particular, we argued that both Hermitian and unitary operators play a major role in the mathematical apparatus of QM. Let us work with those operators a bit more.

1. Show that if two square matrices of the same rank $\hat{A}$ and $\hat{B}$ are related by a unitary transformation $\hat{A} = \hat{U}^\dagger \hat{B} \hat{U}$ then their traces and determinants are the same.
2. Show that all possible expectation values of Hermitian operators $\hat{L}^\dagger \hat{L}$ and $\hat{L}\hat{L}^\dagger$ are non-negative.
3. Show that if $\hat{F}$ is Hermitian then $\hat{G} = exp\left(i\hat{F}\right)$ is unitary.

Problem 2: Complex conjugation (10 pt).

Consider the complex conjugation operator,

$$\hat{K}\psi(x) = \psi^*(x) \tag{1}$$

1. Is this a linear operator? Find the inverse operator $\hat{K}^{-1}$. Does the Hermitian-conjugated operator exist for $\hat{K}$?
2. Find the eigenvalues and eigenfunctions of this operator. Hint: show that $\psi(x) = e^{ia}g(x)$, where $g(x)$ is an arbitrary real function, is an eigenfunction.

Problem 3: Expectation values of coordinate and momentum (10 pt).

A particle is prepared in a state described by the following wave function,

$$\psi(x) = Cexp\left[i\frac{p_0}{\hbar}x - \frac{(x-x_0)^2}{2a^2}\right], \tag{2}$$

where p_0, x_0, and a are some real parameters. Find the probability distribution for finding the particle around x. Find the expectation values and fluctuations of coordinate and momentum. Hint: recall that coordinate operator is $\hat{x}\psi(x) = x\psi(x)$ and momentum operator is $\hat{p}\psi(x) = -i\hbar(d/dx)\psi(x)$.

Problem 4: Dipole moment (10 pt).

An expectation value of the dipole moment of a quantum-mechanical system can be defined as

$$\langle \mathbf{d} \rangle = \int_{-\infty}^{+\infty} \prod_b d^3\mathbf{r}_b \; \psi^*(\mathbf{r}_1, \mathbf{r}_2, ...) \left[\sum_a q_a \mathbf{r}_a\right] \psi(\mathbf{r}_1, \mathbf{r}_2, ...) \tag{3}$$

Show that the expectation value $\langle \mathbf{d} \rangle$ is zero for the states of definite parity, i.e. for the states $\psi(\mathbf{r}_i)$: $\hat{P}\psi(\mathbf{r}_i) = \pm\psi(\mathbf{r}_i)$.