PHY7400. Homework 5

This homework assignment is due on October 12.

Suggested reading:
S. Weinberg, *Lectures on Quantum Mechanics*, Chapters 2-3;

Problem 1: Fun with the square well (10 pt).

Find energy spectrum of a particle confined in the potential
\[
U(x) = \begin{cases} 
\alpha \delta(x), & \text{for } |x| < a \\
\infty, & \text{for } |x| \geq a,
\end{cases}
\]
where \(\alpha > 0\). How does delta-function potential affect the energy levels in this problem compared to the case of ordinary square well potential? In addition

1. show that if \(m \alpha a/\hbar^2 \gg 1\) then the low energy spectrum consists of pairs of closely-spaced (in energy) levels
2. find the equation whose solutions describe energy levels in the potential
\[
U(x) = \begin{cases} 
\alpha \delta(x - b), & \text{for } |x| < a \\
\infty, & \text{for } |x| \geq a,
\end{cases}
\]
for \(b < a\) and show that it reduces to the original equation for \(b = 0\)

Problem 2: Potential step revisited (10 pt).

Consider potential described by a continuous real function \(U(x)\) such that
\[
U(x) = \begin{cases} 
0, & \text{for } x \to -\infty \\
U_0, & \text{for } x \to \infty,
\end{cases}
\]
where \(U_0 > 0\). Show that the transmission coefficient \(T(E) \sim \sqrt{E - U_0} \to 0\) for \(E \to U_0\). Compare to the case of a potential step.

*Hint:* Recall that the transmission coefficient \(T(E)\) is defined as
\[
T(E) = \lim_{x \to -\infty} \frac{|j_R(\psi_R)|}{|j_L(\psi_L)|},
\]
where \(j_L,R\) is the current associated with the way going to the left(right).
Problem 3: Fun with the ladder operators (10 pt).

Consider (differently normalized) ladder operators \( \hat{a} \) for particle in a harmonic oscillator potential discussed in class:

\[
\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}), \quad \hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}),
\]

(5)

where \( \hat{Q} = x(\sqrt{m\omega/\hbar}) \) and \( \hat{P} = \hat{p}(1/\sqrt{\hbar m\omega}) \).

1. Write the harmonic oscillator Hamiltonian in terms of \( \hat{P} \) and \( \hat{Q} \) and in terms of \( \hat{a}^\dagger \) and \( \hat{a} \).

2. Show that for harmonic oscillator eigenfunctions \( \psi_n(x) \)

\[
\hat{a}^\dagger \psi_n(x) = \sqrt{n + 1}\psi_{n+1}(x)
\]

(6)

3. Finally, find all non-zero matrix elements of operators \( \hat{P} \) and \( \hat{Q} \) between different harmonic oscillator wavefunctions \( \psi_n(x) \) and \( \psi_k(x) \). Hint: express \( \hat{P} \) and \( \hat{Q} \) in terms of \( \hat{a} \) and \( \hat{a}^\dagger \).