Advanced Topics in EFT

Homework 1 Solutions

Problem 1:

1. 

\[
\frac{N(N-1)(N-2)}{1 \cdot 2 \cdot 4 \cdot 6} \times \frac{(N+1)N \times (N+2)}{1 \cdot 3 \times [1]} \times \frac{(N+3)}{144} = \frac{N^2(N^2-1)(N^2-4)(N+3)}{144}
\]

\[
\frac{N(N+1)(N+2)}{1 \cdot 2 \cdot 3} = \frac{N(N+1)(N+2)}{6}
\]

2. 

\[
5 \otimes 40 = 175 \oplus 15 \oplus 10.
\]
Working this out gives $24 \otimes 75 = 1024 \oplus 126 \oplus 126 \oplus 175 \oplus 175 \oplus 75 \oplus 75 \oplus 24$.

Notice that this is the product of two real representations, and so the result is real. Either the irreducible representations on the right are real (such as the $1024 = (1,1,1,1)$), or they come in complex conjugate pairs.

**Problem 2:**

To prove $8 \otimes \bar{3} = \bar{15} \oplus 6 \oplus \bar{3}$ in $SU(3)$, we will define:

\[
v^i_j = 8 \quad \xi_i = 3 \quad \text{Note the lower index!}
\]

The product is $v^i_j \xi_k$. To decompose this into irreducible representations of $SU(3)$, remember that we need symmetric and antisymmetric combinations of indices of the same type (upper or lower), and all traces between an upper and lower index must vanish. We have two lower indices, so start by symmetrizing these indices:

\[
v^i_j \xi_k = \frac{1}{2}(v^i_j \xi_k + v^i_k \xi_j) + \frac{1}{2}(v^i_j \xi_k - v^i_k \xi_j) \equiv \frac{1}{2}(S^i_{jk} + A^i_{jk})
\]

$S^i_{jk}$ and $A^i_{jk}$ are symmetric and antisymmetric in the indices $(jk)$, respectively, but they are not traceless in the $(ij)$ and $(ik)$ indices. To accomplish this, we must subtract (and then add back) the traces:
\[ T(a, b)_{jk}^i \equiv a \delta_j^i v_k^l \xi_l + b \delta_k^i v_j^l \xi_l \]

Now take the traces of \( S - T(a, b) \) (and recall that \( v_i^i = 0 \)):

\[
(S - T(a, b))_{ik}^i = (1 - 3a - b)v_k^l \xi_l = 0 \\
(S - T(a, b))_{ji}^i = (1 - a - 3b)v_j^l \xi_l = 0
\]

This has the solution \( a = b = \frac{1}{4} \). Now repeat for the antisymmetric combination:

\[
(A - T(c, d))_{ik}^i = (-1 - 3c - d)v_k^l \xi_l = 0 \\
(A - T(c, d))_{ji}^i = (+1 - c - 3d)v_j^l \xi_l = 0
\]

This has the solution \( c = -d = -\frac{1}{2} \). So we finally have

\[
v_j^i \xi_k = \frac{1}{2} (v_j^i \xi_k + v_k^i \xi_j - \frac{1}{4} \delta_j^i v_k^l \xi_l - \frac{1}{4} \delta_k^i v_j^l \xi_l) \\
+ \frac{1}{2} (v_j^i \xi_k - v_k^i \xi_j + \frac{1}{2} \delta_j^i v_k^l \xi_l - \frac{1}{2} \delta_k^i v_j^l \xi_l) \\
- \frac{1}{8} \delta_j^i v_k^l \xi_l + \frac{3}{8} \delta_k^i v_j^l \xi_l
\]

1. The first line is three symmetric tensors \((6 \times 3 = 18)\), and there are three traceless conditions, one for each value of the remaining free index \((-3)\). So this line has a total of 15 components. Furthermore, it has more lower indices than upper indices, therefore this is a \(15\).

2. The second line is three antisymmetric tensors \((3 \times 3 = 9)\), and there are again three traceless conditions \((-3)\). Therefore there are 6 components. The \(6 = \) \(\square\) is symmetric in its indices, as is the \(\bar{6} = \) \(\square\). To manifest this, we can write the second line as

\[
\frac{1}{4} \varepsilon_{jkl} (\varepsilon^{lmn} v_m^l \xi_n + \varepsilon^{imn} v_m^i \xi_n)
\]

This is symmetric in upper indices \((il)\), and so this is the \(6\).

3. The final line has only one free index (indices on Kroneker deltas don’t count, since these are invariant tensors that do not transform). Furthermore, the free index is a lower index, so it is a \(3\).

Thus we have shown \(8 \otimes 3 = 15 \oplus 6 \oplus \bar{3}\). Now aren’t you glad Young tableaux make all this nonsense unnecessary?!

Problem 3:
Now I ask you to do some physics (but only a little, I promise!) and consider the quantum $D = 3$ simple harmonic oscillator. To make life much easier, I’m going to work in units where $\hbar = m = \omega = 1$. You can feel free to reinsert units where and when appropriate.

1. The SHO in three dimensions has a Hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^{3} (p_k^2 + q_k^2) = \sum_{k=1}^{3} a_k^\dagger a_k + \frac{3}{2}$$

where $a = \frac{1}{\sqrt{2}}(q - ip)$. It is immediately clear from the second form of the Hamiltonian that there is an $SU(3)$ symmetry of the form $a \to U a = (1 + i\beta^a T^a) a$, where the $SU(3)$ indices are the spacial dimensions (actually, it’s a $U(3)$ symmetry, but we won’t worry about the phase). This is quite a fascinating symmetry, since it is more than just ordinary rotations, which would be $O(3)$. In a later part of this problem you will prove that this rotation symmetry is a subgroup of this larger symmetry.

To find the conserved charges, we employ Noether’s Theorem, which requires the use of the Lagrangian:

$$L = \frac{1}{2} \sum_{k=1}^{3} (\dot{q}_k^2 - q_k^2)$$

How do the coordinates and velocities transform under this symmetry? If we take the transformation law for $a$, along with the definition of $a$ in terms of $q$ and $\dot{q} \equiv p$, we can easily derive by matching real and imaginary parts:

$$\delta q = \beta^a T^a \dot{q}$$
$$\delta \dot{q} = -\beta^a T^a q = +\beta^a T^a \ddot{q}$$

where I used the equation of motion $\ddot{q} = -q$ in the last line. Notice that this transformation law is integrable: $\delta \dot{q} = d(\delta q)/dt$. Now it is pretty straightforward to work out the change in the Lagrangian, and sure enough, you find $\delta L = \beta^a \frac{dF^a}{dt}$, where

$$F^a \equiv \dot{q}T^a \dot{q} - qT^a q$$

Finally, we can write down the Noether current from this transformation law. In $0 + 1$ dimensions (that is, Quantum Mechanics!) the current is the same thing as the charge and so we have$^1$:

$$Q^a \equiv \frac{\partial L}{\partial \dot{q}} \delta^a q - F^a = + (\dot{q}T^a \dot{q} + qT^a q)$$
$$= (q + ip)T^a(q - ip)$$
$$= a^\dagger T^a a$$

It is a very straightforward exercise using $[a_i, a_j^\dagger] = \delta_{ij}$ to show that these objects do indeed transform in the adjoint representation of $SU(3)$:

$$[Q^a, Q^b] = i f^{abc} Q^c$$

and that they commute with the Hamiltonian, as a good symmetry should.

$^1$I use the notation $\delta q \equiv \beta^a \delta^a q$. Also notice that the commutator of $q$ and $p$ is irrelevant here, since $\text{Tr} T^a = 0$. 

2. The ground state is invariant under these $SU(3)$ transformations, since $a\ket{0} = 0$. Furthermore:

$$[Q^a, a_l] = a_l^T a^a_{ij} [a_j, a^\dagger_l] = a_l^T a^a_{il}$$

$$[Q^a, a_i] = [a^\dagger_l, a_i] T^a_{ij} a_j = -T^a_{ij} a_j = a_i [-T^*]^a_{il}$$

Therefore $a_l^\dagger$ transforms in the $\bar{3}$, while $a_i$ transforms in the $\bar{3}$. All states of definite energy are products of $a_l^\dagger$ on the vacuum state, thus the $n$ level is a product of $n$ $3$s. Furthermore, since these operators commute, the result is symmetric in the $SU(3)$ indices. The Young tableau for such a representation is a single row of $n$. By the Young tableau rules for $SU(3)$ we can then read off:

$$\dim\{|n\rangle\} = \frac{[3 + (n - 1)]!}{n! \cdot 2} = \frac{(n + 1)(n + 2)}{2}$$

Of course, this is exactly the right answer, if you look it up in your favorite undergraduate QM textbook!

3. We now want to realize the angular momentum and show that it is a subgroup of the $SU(3)$ symmetry.

$$L_i = \varepsilon_{ijk} q_j p_k = -\frac{i}{2} (a_j + a_j^\dagger)(a_k - a_k^\dagger)$$

$$= \frac{i}{2} \varepsilon_{ijk} (a_j a_k^\dagger - a_j^\dagger a_k)$$

$$= -i \varepsilon_{ijk} a_j^\dagger a_k$$

where I have used the asymmetry of the Levi-Civita symbol and the commutation relations of the creation and annihilation operators. Armed with this formula, we can write:

$$L_1 = -i(a_2^\dagger a_3 - a_3^\dagger a_2) = a^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} a = 2Q^7$$

$$L_2 = -i(a_3^\dagger a_1 - a_1^\dagger a_3) = a^\dagger \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} a = -2Q^5$$

$$L_3 = -i(a_1^\dagger a_2 - a_2^\dagger a_1) = a^\dagger \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} a = 2Q^2$$

I have used $T^a = \frac{1}{2} \lambda^a$ where $\lambda^a$ are the Gell-Mann matrices. So indeed, the angular momentum is given by $L = (2Q^7, -2Q^5, 2Q^2)$. You can feel free to check that this set of generators does indeed form an $SU(2)$ subgroup.
4. Now we add a term to the Hamiltonian $\Delta H = \alpha L^2$, where presumably $\alpha$ is small so we can do perturbation theory. This term explicitly breaks the $SU(3)$ symmetry since it picks out directions $(2,5,7)$ in our $SU(3)$ root space. But it does not break the $SU(2)$ symmetry; since the operator is just the Casimir operator of $SU(2)$, which is proportional to the identity. Therefore we know (without doing any work!) that the energy levels will split based on the size of their total angular momentum.

The first four energy levels are (in $SU(3)$ notation) the $1, 3, 6, 10$, corresponding to $n = 0, 1, 2, 3$. To understand the angular momenta of these states without doing any work (!), remember that the $n$ level has $n$ symmetric indices, and therefore should contain a spin-$n$ state. Furthermore, since all the states are made out of vector objects (spin-1), there can be no half-integer spin states (or if you wish, the theory is bosonic). From this it is very easy to work out the spin states from the Clebsch-Gordan decomposition. The relevant information is given in the table, and the energy diagram for the first four energy levels is shown below. The angular momentum degeneracy tells you how many states remain degenerate after $\Delta H$ is turned on.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$SU(3)$</th>
<th>$2l + 1$</th>
<th>$\Delta H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>$2\alpha$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>5 + 1</td>
<td>$6\alpha, 0$</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>7 + 3</td>
<td>$12\alpha, 2\alpha$</td>
</tr>
</tbody>
</table>