PHY7410. Homework 1

This homework assignment is due on **January 23**.

Suggested reading:

L. Landau and E. Lifshitz, *Quantum Mechanics*, Chapter 6.

Problem 1: Warm-up: square well with an extra dimension.

As you might know, string theory requires extra space dimensions to be present in the theory in order for it to be consistent. Where are those “extra dimensions”? String theorists like to say that some of them are “compactified” and can only be seen with very high energy probes. Incidentally, simple quantum mechanics of single particle in the infinite square well allows to model this phenomenon in a very simple manner.

1. Find energy eigenvalues and normalized eigenstates of a particle in a one-dimensional square well, i.e. with the potential

$$U(x) = \begin{cases} 0, & \text{for } x \in (0, a) \\ \infty, & \text{for } x \notin (0, a) \end{cases} \tag{1}$$

   Since wave function vanished outside the interval $(0, a)$ we can say that our particle “lives on a segment.”

2. Now consider the two dimensional case (x, y). Let the x coordinate be the segment $(0, a)$ where particle lives (as above). Assume “periodic boundary conditions” for the y coordinate, i.e. make the identification

$$(x, y) \sim (x, y + 2\pi R), \tag{2}$$

   which is equivalent to the case of a quantum particle living on a cylinder of radius R and height a (this procedure is called “compactification” of y-dimension). Solve Schroedinger equation in two dimensions (with $U(x, y) = 0$ on the cylinder) and write the energy spectrum for the problem. You don’t need to normalize the solution. Hint: wave function will also be periodic in y: $\psi_{k,l}(x, y) = \psi_{k,l}(x, y + 2\pi R)$, where k is a wave number for x-coordinate and l is a wave number for y-coordinate.

3. Observe that the new spectrum contain all "one-dimensional" energies (see part 1) for the case $l = 0$. Consider a very long and thin cylinder, $R \ll a$, and find energy of the lowest "new" eigenstate. Compare its energy to the energy spectrum of a particle on a segment to show that only probes of high energy will be able to detect the presence of compactified extra dimension (with the low-energy part of the spectrum totally identical to the case discussed in part 1). Notice that it is the fact that a long thin cylinder looks like a one-dimensional object that led string theorists to build self-consistent string models.

Problem 2: Oscillator in electric field.

Determine the eigenfunctions and energy spectrum for one-dimensional harmonic oscillator placed in the external homogeneous electric field. To do that

1. Recall that the charged particle placed in the external homogeneous electric field acquires potential energy

$$\Delta U(x) = -q\mathcal{E}x$$

   and determine the *exact* eigenvalues and eigenfunctions of charged one-dimensional harmonic oscillator ($U(x) = kx^2/2$) placed in the electric field $\mathcal{E}$. In addition, find the *electric polarizability* β of harmonic oscillator. As you might know, electric polarizability defines energy shift which is quadratic in the applied field, i.e. $\Delta E = -\beta\mathcal{E}^2/2$.
2. Now assume that the electric field is weak, so that we can apply *perturbation theory*. Determine first two corrections to the energy spectrum. Compare your result to the exact result discussed in part 1.

Problem 3: Square well in electric field.

Determine first and second-order corrections to energy of charged particle placed in the infinite square well. Find electric polarizability β of energy eigenstates.