PHY7410. Homework 3

This homework assignment is due on **February 17**.

Suggested reading:

L. Landau and E. Lifshitz, *Quantum Mechanics*, Chapters VIII-IX.

Problem 1: Born-Oppenheimer Approximation.

Born-Oppenheimer approximation (BOA) is used in many problems related to multi-body calculations in quantum mechanics, atomic physics, and more recently, in particle physics to describe the so-called "exotic" states.

Consider a solvable toy model for a "molecular state" (like H_2 ion), containing one light particle ("electron") of mass m connected to two heavy particles ("protons") with mass M. For the sake of this model, let's assume that the interaction potential resembles that of a string with natural length a and spring constant β. The Hamiltonian is

$$H = \frac{p_1^2}{2M} + \frac{p_2^2}{2m} + \frac{p_3^2}{2M} + \frac{\beta}{2}\left[(x_2 - x_3 - a)^2 + (x_2 - x_1 + a)^2\right] \qquad (1)$$

1. Fix the distance between the two heavy particles to be R, and find the energy spectrum $E_n(R)$ of the light particle as a function of R. Hint: to accomplish that, switch to the center-of-mass coordinates of a two-"proton" system R_{CM} and R.
2. Now treat $E_n(R)$ as a potential between the protons, and solve for the proton energies E_{mn}. Do the calculation in the heavy particles' center of mass frame. The result is the Born-Oppenheimer approximation for this problem.
3. Solve the problem exactly and find the energy levels for this problem. Based on this, estimate the accuracy of Born-Oppenheimer approximation. Show that the error is small for $m \ll M$.

Problem 2: Fun with spin.

Let us get acquainted with spin formalism more closely.

1. Find eigenvalues and eigenfunctions of the operator

$$\hat{f} = a + \vec{b} \cdot \hat{\vec{\sigma}},$$

where a and $\vec{b}$ are arbitrary scalar and vector parameters.

2. Using the properties of Pauli matrices, find explicit form of the operator

$$\hat{F} = F(a + \vec{b} \cdot \hat{\vec{\sigma}}),$$

where $F(z)$ is an arbitrary "good" function of variable z.

3. Consider a particular case of a rotation operator described in class,

$$\hat{F} \equiv \hat{G}(\phi_0) = exp\left(\frac{i}{2}\phi_0 \vec{n} \cdot \hat{\vec{\sigma}}\right),$$

which transforms a spin wave function $\psi' = \hat{R}(\phi_0)\psi$. Find $\psi_{s_n=\pm 1/2}$ eigenfunctions of this operator for spin projection onto some arbitrary quantization axis $\vec{n}$.

4. Using these results, prove that the products of two spinors

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \text{ and } \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$S = \Phi^\dagger \Psi$ and $V = \Phi^\dagger \hat{\sigma}_i \Psi$, transform respectively as a scalar and vector under rotations.