Problem 1: The Born approximation and the Optical Theorem

An particle of mass \( m \), moving in three dimensions, is being scattered by the radial potential \( V(\mathbf{r}) \equiv V(r) \). Set \( \hbar = 1 \) in this and the following problem.

(i) Show that the scattering amplitude \( f^{\text{Born}}(\theta) \)

\[
f^{\text{Born}}(\theta) = -\frac{m}{2\pi} \langle \phi_f | \hat{V} | \phi_i \rangle = -\frac{m}{2\pi} \int d^3 \mathbf{r} V(\mathbf{r}) e^{i(\mathbf{p}_i - \mathbf{p}_f) \cdot \mathbf{r}}
\]  

is real, in seeming contradiction with the optical theorem that relates \( \text{Im} f(0) \) to the total cross section. It is actually ok, as the cross-section is of the second order in the potential. Thus, in order to check the optical theorem we have to go to higher orders in perturbation theory.

(ii) Evaluate the scattering amplitude to second order in \( V \), i.e. show that

\[
f^{(2)}(\theta) = -\frac{m}{2\pi} \left[ \langle \phi_f | \hat{V} | \phi_i \rangle + \sum_j \frac{\langle \phi_f | \hat{V} | \phi_j \rangle \langle \phi_j | \hat{V} | \phi_i \rangle}{E_i - E_j + i\epsilon} \right]
\]

Evaluate \( \text{Im} f^{(2)}(0) \) to verify that the optical theorem is satisfied, provided that the Born cross section is used. This problem shows the utility of the optical theorem in evaluating the effects of higher order perturbative corrections to transition amplitudes.

Problem 2: Gauge invariance

One of the fundamental principles of modern theories of particle interactions is the principle of gauge invariance. This principle states that phys-
ical properties of a theory would not change if we perform a coordinate-
dependent change of the phase of a wave function, while simultaneously
changing (“gauging”) the vector potential. For electrodynamics it implies
that physics does not change if we substitute

$$\vec{A}(r) \rightarrow \vec{A}(r) + \vec{\partial} \chi(r),$$

(3)

where $\chi(r)$ is an arbitrary scalar function. Let’s check that it works in
quantum mechanics. To do that, consider scattering of charged particle off
magnetic field $\vec{B}(r)$.

1. Using the potential written in terms of a vector potential, obtain the
   Born approximation scattering amplitude as a function of Fourier trans-
   form of a vector potential.
2. Prove that neither the scattering amplitude nor the cross section change
   under the gauge transformation of the vector potential of Eq. (3).

**Problem 3: Phase shifts in Born approximation**

Derive an expression for the phase shift $\delta_l$ in the leading Born approximation.
To achieve that, write the formula for the Born amplitude (in particular,
the term $\sin(qk)/(qr)$) as an expansion over $P_l(\cos \theta)$ and then compare the
obtained result to Faxen-Holtsmark formula. Hint: note that

$$\frac{\sin z}{z} = J_{1/2} \left( \sqrt{\frac{\pi}{2z}} \right).$$

(4)