PHY8850. Homework 2

This homework assignment is due on **October 4**. The maximum possible score of this homework, if not turned in by 5 pm that day, will be linearly decreased $N = N_{max}(1 - 0.2n)$, where n is the number of days.

Suggested reading:

M. Peskin and D. Schroeder, *"An Introduction to QFT,"* chapters 2 and 3.

Problem 1: Dilatation symmetry.

In class, we discussed various symmetries and conserved quantities that follow from those symmetries. Let us consider one more example that could be interesting for the future studies of QCD. Consider infinitesimal scale transformations (a.k.a. dilatations):

$$\delta x^\mu = \alpha x^\mu, \ \delta\phi(x) = D\alpha\phi(x), \tag{1}$$

where α is infinitesimally small and D is some constant. Let us see if this transformation is indeed a symmetry of a classical massless real scalar field theory given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \tag{2}$$

(i) Show that the action for the real scalar field $\phi(x)$ is dilatation-invariant in four space-time dimensions provided that $D = -1$

(ii) Identify the Noether current associated with this symmetry and use equations of motion to check that its divergence vanishes.

Problem 2: Zero-point energy.

Show that the vacuum expectation value of the scalar field Hamiltonian discussed in class is given by

$$\langle 0|H|0\rangle = A\pi m^4 \delta^{(3)}(0)\Gamma(-2), \tag{3}$$

where $\Gamma(x)$ is a gamma-function. Determine the constant A.

As you can see, this expression is the product of two divergent terms. Note that the *normal ordering* gets rid of this c-number divergent term.

Problem 3: Energy, momentum and charge.

As was mentioned in class, charged scalar particles can be described by a *complex* scalar field,

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{ikx} \right), \\ \phi^\dagger(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(b(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx} \right), \end{aligned} \tag{4}$$

where $a(\vec{k})$ and $b(\vec{k})$ are annihilation operators and $a^\dagger(\vec{k})$ and $b^\dagger(\vec{k})$ are creation operators.

Calculate the operators of energy $: H :$, momentum $: \vec{P} :$, and charge $: Q :$ of a complex scalar field in terms of creation and annihilation operators (note the normal ordering).