PHY8850. Homework 7

This homework assignment is due on April 11. The maximum attainable score for this homework, if not turned in by 5 pm that day, will be linearly decreased \( N = N_{\text{max}}(1 - 0.2n) \), where \( n \) is the number of days.

Suggested reading:
M. Peskin and D. Schroeder, “An Introduction to QFT,” chapters 4-5; or D. Bailin and A. Love, “Introduction to Gauge Field Theory,” chapters 4-6.

Problem 1: Two neutrino Phase Space in d-dimensions

In muon decay, the lifetime calculation involves integrating over phase space of the two final state neutrinos. Let us make the problem more interesting by considering this problem in d-dimensions (it turns out that changing the number of dimensions helps to regularize, i.e. subdue divergences, of integrals in multiloop computations). Consider the phase space integral in d-dimensions:

\[
I^{\alpha\beta} = \int \frac{d^{d-1}q_2}{2E_2} \frac{d^{d-1}q_4}{2E_4} q_2^\alpha q_4^\beta \delta^{(d)}(Q - q_2 - q_4)
= A(Q) \frac{Q^\alpha Q^\beta}{Q^2} + B(Q) \frac{Q^2 g^{\alpha\beta}},
\]

(1)

where the second line follows from covariance. Since the neutrinos are (almost) massless, all you need to know is that

\[
\int d^{d-1}q_2 = \Omega_{d-1} \int E_2^{d-2}dE_2,
\]

(2)

where

\[
\Omega_d = \int_0^{\pi} d\theta_{d-1} \sin^{d-2} \theta_{d-1} \cdots \int_0^{\pi} d\theta_2 \sin \theta_2 \int_0^{\pi} d\theta_1 = \frac{2\pi^{d/2}}{\Gamma(d/2)}
\]

is the angular integral in d dimensions. Work in the muon rest frame, \( Q = (Q^0, \vec{0}) \).

(a) Study the case \( \alpha = \beta = 0 \) to obtain \( A + B \), expressed in terms of \( \Omega_{d-1} \), powers of 2 and powers of \( Q^0 \).

(b) Study the case \( \alpha = i, \beta = j \) \( (i, j = 1, 2, 3) \) to obtain an expression for \( B \). It might help you to use

\[
\int d\Omega_{d-1} q_2^i q_4^j = C q_2^i \delta^{ij},
\]

(3)
where you should determine $C$.

(c) Combine your results from parts (a) and (b) to obtain $I^{\alpha\beta}$ expressed in terms of $\Omega_{d-1}$, powers of 2 and powers of $Q^2$. Obtain the $d = 4$ limit of your answer.

Problem 2: Fermion projection operators

As you might remember, a solution of the Dirac equation for a free particle was written as

$$u(p, s) = \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \bar{\sigma} \xi^s}}\right), \quad v(p, s) = \left(\frac{\sqrt{p \cdot \sigma} \eta^s}{-\sqrt{p \cdot \bar{\sigma} \eta^s}}\right), \quad (4)$$

where $\xi^s$ and $\eta^s$ are Pauli spinors that satisfy a completeness relation,

$$\sum_s \xi^s \xi^{s\dagger} = \sum_s \eta^s \eta^{s\dagger} = 1, \quad (5)$$

and $\sigma^\mu = (1, \bar{\sigma})$, $\bar{\sigma}^\mu = (1, -\bar{\sigma})$, where $\bar{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ are the usual Pauli matrices. In the calculations of unpolarized cross-sections we will be using the so-called Casimir trick, which involves calculations of polarization sums. Let us derive some of those polarization sums.

(a) Using the solution above, prove that

$$\sum_{s=1}^{2} u(p, s)^* u(p, s) = (\not{p} + m) \equiv \Lambda_+,$$

$$\sum_{s=1}^{2} v(p, s)^* v(p, s) = (\not{p} - m) \equiv \Lambda_- \quad (6)$$

(b) Show that $P_\pm = \Lambda_\pm/(2m)$ are projection operators, i.e. they satisfy $P_\pm^2 = P_\pm$, and $P_+ P_- = 0$. How do those projectors act on basic spinors $u(p, s)$ and $v(p, s)$?