Stochastic multiplicative processes for financial markets

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Abstract

We study a stochastic multiplicative system composed of finite asynchronous elements to describe the wealth evolution in financial markets. We find that the wealth fluctuations or returns of this system can be described by a walk with correlated step sizes obeying truncated Lévy-like distribution, and the cross-correlation between relative updated wealths is the origin of the nontrivial properties of returns, including the power-law distribution with exponent outside the stable Lévy regime and the long-range persistence of volatility correlations.

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1. Introduction

Multiplicative processes have been well studied in different contexts and widely applied to various research fields, such as the biological, social, and economic systems [1–6]. One of the major interests in these processes is the generation of power laws [1,3–5,7], which have been observed in many natural domains and indicate the properties of scale invariance and universality. In fact, the investigations of power-law behaviors in systems with stochastic multiplicative dynamics can be traced back to decades, and the underlying physical mechanisms are still to be understood.
For the applications in financial problems, the assumption of multiplicative property for the individual capital investments $w_i$ (the index $i = 1, \ldots, N$ may correspond to various investors/traders or companies (stocks) in the market) leads to

$$w_i(t + 1) \sim \lambda(t)w_i(t)$$

(1)

appearing in the dynamics of wealth/capital evolution. That is, the individual capital $w_i$ at time $t + 1$ is proportional to the invested capital itself. The random factor $\lambda(t)$ reflects the relative gain/loss incurred by individuals between time $t$ and $t + 1$, and is chosen from a probability distribution $\Pi(\lambda)$.

The Pareto power-law distribution [8] of individual wealths $w_i$,

$$P(w) \sim w^{-1 - z_w}$$

(2)

has been found in many of the previous studies of multiplicative processes [1,3–7], which all follow the above dynamics (1) (plus some additional crucial dynamical elements that we will discuss later), and been verified repeatedly in the last hundred years in most of the capitalistic societies, e.g., for individual income and wealth [8,9], size of business firms [10], etc. In spite of the significant fluctuations in the total wealth (with occasional spectacular booms and crashes), the exponent $z_w$ was observed typically in the range $1 < z_w < 2$, i.e., within the stable Lévy regime [11].

Besides the above distribution property of individual wealth, we are more interested in studying the relative fluctuations of the total wealth $W(t) = \sum_{i=1}^{N} w_i(t)$, which may represent the capitalization (total market value) of a company, or the market price of a stock (when normalized to the total number of shares that stock has on the market), or the market index. The fluctuations of $W(t)$ over arbitrary periods of time $\tau$ define the market returns $r(\tau)$ during these time periods:

$$r(\tau) = \ln W(t + \tau) - \ln W(t) ,$$

(3)

which have attracted much attention in recent financial studies, in particular for their statistical properties [12,13]. There are some empirical features generic for different financial markets, including:

- The power-law decay of probability density of returns $r$

$$P(r) \sim r^{-1 - z} ,$$

(4)

in tail region, but with exponent $z > 2$ different from that of wealth distribution (2), i.e., outside the Lévy regime [14–16]. For larger $r$ values, $P(r)$ can be fitted by an exponential in some observations [17–19].

- The very short range (about a few minutes) correlation of returns and long-range persistence for correlation of square or absolute value of returns (the so-called volatility clustering) [12,13,15,17].

It is interesting to not only reproduce these stylized facts through microscopic models, but also understand the intrinsic mechanisms dominating the processes.
2. Generalized Lotka–Volterra system

The framework we use to describe the dynamics of financial systems is the Generalized Lotka–Volterra (GLV) model proposed by one of us a few years ago [3,4]:

\[ w_i(t+1) = \lambda(t) w_i(t) + a \bar{w}(t) + c(w_1,w_2,\ldots,w_N,t) w_i(t) \] (5)

with the average wealth \( \bar{w}(t) = W(t)/N \). The multiplicative property at the individual level, i.e., Eq. (1), is expressed in the first r.h.s. term, and the second term represents the property of wealth at the social level, which may correspond to the social security, subsidies, or funded services, and can also be interpreted as arising from the diffusion of wealth between agents (by services, taxes, etc.). This term is set to be proportional to the average wealth \( \bar{w} \) with an important coupling parameter \( a \), supplying the correlation and coupling between investors or companies that is crucial for the distribution properties of wealth and wealth fluctuation shown below. Generally, the last term \( c w_i \) which corresponds to the competition in the market does not qualitatively affect the properties of the model [20]. For simplicity, we set it as 0 in the present study. Therefore, we have

\[ w_i(t+1) = \lambda(t) w_i(t) + a \bar{w}(t) . \] (6)

Note that in this stochastic multiplicative system an asynchronous updating mechanism [3,4,21–23] is used, due to the fact that in a real market the capitals of different agents or companies are not updated simultaneously. Thus, at each discrete time \( t \), one of the elements \( i \) is chosen randomly and updated according to Eq. (6), while the other \( w_i \)'s are kept unchanged. Moreover, the random factor \( \lambda \) is taken in a rather narrow range around 1, since in practice the price returns for very small time intervals are usually rather small.

3. Power-law behaviors and cut-off effect

The system described by Eq. (6) does not approach a steady state; instead the total wealth grows (or decays) exponentially with superimposed fluctuations, i.e., \( W(t) \sim \exp(\kappa t/N) \) with \( \kappa \) depending on \( a \) as well as the mean and standard deviation of \( \lambda \) distribution. Nevertheless, the instantaneous values of wealth (or the relative value \( w_i(t)/W(t) \)) fulfill the Pareto power-law distribution (2), with exponent \( \alpha_w < 2 \) [21], that is, inside the stable Lévy regime.

More interesting results for this multiplicative system are obtained by studying the wealth fluctuations (i.e., returns) \( r(t) \) defined by Eq. (3). (Note that due to the growth effect of total wealth \( W \sim \exp(\kappa t/N) \) in this model, the return in (3) should be detrended, that is, change by a constant: \( r(t) \rightarrow r(t) - \kappa t/N \).) For a certain time interval \( \tau \), the return \( r(t) \) can be written as \( \sum_{s=t}^{t+\tau-1} \ln[(W(s+1) - W(s))/W(s) + 1] = \sum_s \ln[(\lambda(s) - 1)w_i(s)/W(s) + a/N + 1] \) according to Eq. (6). \(^1\) Since the relative value of wealth \( w_i/W \) at each step is small and generally \( \lambda - 1 \) is in a narrow range around 0,

\(^1\) Note that at each time step \( s \) the label \( i \) of the updated wealth \( w_i \) changes and is actually \( i(s) \).
one obtains that the properties of the (detrended) return \( r(\tau, t) \) are the resultant of \( \tau \) steps walk:

\[
R(\tau) = \sum_{s=\tau}^{t+\tau-1} (\hat{\lambda}(s) - 1)w_i(s)/W(s)
\]

with step sizes being governed by the power-law distribution (2) with cut-off effect.

For \( \tau=1 \) the distribution of returns differs from that of the relative wealth \( w_i(t)/W(t) \) only by a random factor \( \hat{\lambda}(t) - 1 \), and then displays a similar power-law behavior (4) with exponent \( \alpha \sim \alpha_w \) and a sharp peak around 0. As \( \tau \) increases, the consequence of \( \tau \) step walks described by Eq. (7) is the smoothening of this tip.

Note that for any finite \( N \) the distribution of \( w_i(t)/W(t) \) is not a perfect power-law even if one assumes that of \( w_i \) to be so, since this distribution is truncated from above: \( w_i/W < 1 \) shown as a bending in a log–log plot for very large values of \( w_i \) [24]. We have proven [22] that this upper cut-off effect leads to an exponential decay for extremely large values of returns and for very wide range of interval \( \tau \).

Thus, the process described by Eq. (7) looks like a truncated Lévy walk leading to similar power laws for different \( \tau \) and slow convergence to Gaussian [25]. However, our studies below indicate a more complicated situation, that is, this process is different from the ordinary Lévy walk due to the coupling between step sizes.

The numerical results of Eq. (6) for the distribution of returns \( r(\tau, t) \) (3) are shown in Figs. 1 and 2, where the random gain/loss factor \( \hat{\lambda} \) has a uniform distribution in the range 0.95 < \( \hat{\lambda} \) < 1.05 and the coupling parameter \( a = 0.0002 \) with the corresponding power-law exponent of wealth distribution \( \alpha_w \) being about 1.5 as in usual financial applications. (In all the numerical simulations of this paper the results are calculated after \( t = 10^5N \) updatings, so that the returns series are stationary after detrending the wealth growth factor.) As expected according to the above analysis, the semi-log plots of distribution in Fig. 1(a) exhibit the smearing out of the sharp peak for small time intervals into a dome-like shape for large one, and one obtains the exponential-type behavior for extremely large \( r \) due to the cut-off effect of finite \( N \) system, even for very large interval \( \tau \), which has been observed in some empirical studies [17–19].

The power-law scaling region described by Eq. (4) can be found in the log–log plots of the distribution (see Figs. 1(b) and 2), and interestingly, exponents much larger than that of the wealth distribution (2) are obtained. For small interval \( \tau \), the exponent \( \alpha \) is within the stable Lévy regime, i.e., \( \alpha_w < \alpha < 2 \), while for large intervals (\( \tau > N \)) we obtain the result consistent with the recent empirical observations for both the stock index (German DAX [14] and S&P 500 [15]) and individual stocks [16], that is, the effective exponent \( \alpha > 2 \) outside the Lévy regime. Similar to the findings of real markets [14–16], the exponent increases very slightly with the increase of the time interval, due to the very slow convergence to Gaussian, as shown in Figs. 1 and 2.

The extension of this power-law region with exponent \( \alpha > 2 \) depends on the size \( N \) of the multiplicative system. By comparing the results in Fig. 1 for \( N = 500 \) with that in Fig. 2 for \( N = 5000 \), one can find that the range of the power-law scaling is longer for larger systems. This phenomenon is attributed to the exponential truncation effect discussed above for finite \( N \), which appears more obvious for a smaller system. As observed in the log–log plots, the deviation and bend-down from a straight line
Fig. 1. The probability distribution of returns $r(\tau)$ defined by Eq. (3) for $N = 500$ and different time intervals $\tau$ from $N$ to $5000N$. $\lambda$ uniformly distributes between 0.95 and 1.05, and $a = 0.0002$. The results are averaged over 1500 runs for $\tau \leq 1000N$ and 600 runs for $\tau = 5000N$. (a) Semi-log plot, where for the largest $\tau (=5000N)$ Eq. (7) is used; (b) log-log plot for positive returns.
Fig. 2. Log–log plot of the probability distribution of positive returns $r(\tau)$ (averaged over 100 runs) for system size $N = 5000$ and different time intervals $\tau = N$, 5$N$, 50$N$, and 1000$N$.

occurs for large $r$, which has been found very recently in the Hang Seng Index (HSI) of Hong Kong [17] if one skips the data of first 20 min in daily opening. A similar effect of finite size $N$ has been found in some microscopic models, in particular the Cont–Bouchaud percolation-type model [26], where the $\alpha > 2$ power-law exponent is obtained over an intermediate region which becomes infinitely long for infinite large market size and computer time, while the asymptotic return distribution is expected to be a (stretched) exponential decay in the tails.

However, different from the percolation model, the exponent $\alpha$ for large interval $\tau$ decreases with the increasing size $N$ in our system. As shown in Figs. 1(a) and 2, $\alpha \sim 2.7$ for $N = 500$, and $\alpha \sim 2.3$ for $N = 5000$. Moreover, for $N = 10^5$ (not shown here) $\alpha \sim 2$, which is the border of the stable Lévy regime. Therefore, the property of returns distribution is size dependent, and one can obtain that the effective $N$ for real market is not very large according to the high empirical $\alpha$ value.

Although the value of the exponent $\alpha_w$ for wealth distribution depends on the parameters of the system, the behaviors of the $\alpha > 2$ power-law scaling for return distribution can exist for different parameters (e.g., coupling parameter $a$ and random factor $\lambda$) corresponding to $1 < \alpha_w < 2$. When increasing the value of $a$, one still obtains the $\alpha > 2$ power-law behaviors with larger $\alpha$ value and longer extension for the same $N$. In Fig. 1, $N = 500$ and $a = 0.0002$ (with the corresponding $\alpha_w$ about 1.5), we have $\alpha \sim 2.7$ but with a rather limited range of power-law regime, while for larger $a = 0.0003$ as well as the same $N$ and range of $\lambda$ (with $\alpha_w$ about 1.7) $\alpha = 3.1$ is obtained, and more
importantly, the power-law regime is much longer (close to 3 orders of magnitude), as shown in Fig. 3.

4. Properties of correlations: correlated walk steps and volatility

These results of $\alpha > 2$ are nontrivial, and different from the expectation that the wealth $w_i$ and the fluctuation $r$ should have tails obeying similar power law, since a random walk similar to Eq. (7) with steps of sizes dominated by Lévy-like distribution with finite variance still leads to a behavior within the stable Lévy regime before the crossover to a Gaussian process [25]. However, this expectation is only valid for the random walk with statistically independent step sizes, while for the multiplicative process (6), there are cross-correlations between the relative wealths $w_i(s)/W(s)$ comprising the returns in Eq. (7) at each micro-step $s$. This can be verified by studying the properties of time series $w_i(t)/W(t)$, where $w_i$ refers to the newly updated wealth at each time $t$. As expected, the probability distribution of the walk process of Eq. (7) is almost indistinguishable from that of return $r(\tau)$ shown in Figs. 1–3. However, if randomizing the time series of $w_i(t)/W(t)$, which corresponds to eliminate all the possible correlations but keep the distribution form of step sizes, and then calculating the distribution of $R(\tau)$ due to Eq. (7), one obtains the completely different results, that is, the convergence to Gaussian is faster and only the power-law regions with
Lévy-like exponent $\alpha < 2$ can be found, similar to the process of ordinary truncated Lévy flight [25].

To directly calculate the cross-correlations between the relative updated wealths $w_i(t)/W(t)$, we use the autocorrelation function of

$$\text{Corr}(T) = \frac{\langle x(t)x(t+T) \rangle - \langle x(t) \rangle \langle x(t+T) \rangle}{\langle x^2(t) \rangle - \langle x(t) \rangle^2}$$

for some variable $x(t)$. Although there is no correlation of relative wealths ($x = w_i/W$ in (8)), the nonzero and positive correlation of the square $(w_i/W)^2$ is found with long persistence, as shown in Fig. 4, verifying the existence of the coupling between step sizes of the walk (7) for this multiplicative system. These correlations are attributed to the social term $aw$ introduced in our system (6), supplying the coupling between different wealths. A larger value of $a$ leads to an enhancement of cross-correlations, and then of $\alpha > 2$ behavior, including the value of $\alpha$ and the extension of power-law region (see Fig. 3). Even when keeping the value of $\alpha_w$ for wealth, the correlation of $(w_i/W)^2$ increases with larger $a$. In the inset of Fig. 3, different values of coupling parameter $a$ and different range of $\lambda$ are chosen so that the corresponding $\alpha_w$ keeps the same value 1.5, and from bottom to top the correlations increase with the increasing values of $a$. 

Fig. 4. Correlations of $(w_i/W)^2$ and $R^2(\tau = 1)$ for $N = 500$, $a = 0.0002$, and $\lambda$ in the range (0.95,1.05) (averaged over 1100 runs). Inset: correlations of $(w_i/W)^2$ (averaged over 1000 runs) for $N = 500$ and different $a$ values and $\lambda$ ranges: $a = 0.0008$ and $\lambda \in (0.9,1.1)$ (circles), $a = 0.0002$ and $\lambda \in (0.95,1.05)$ (stars), and $a = 5 \times 10^{-5}$ and $\lambda \in (0.98,1.02)$ (triangles), which all correspond to the same exponent $\alpha_w = 1.5$ of wealth distribution, as well as very small $a = 2 \times 10^{-6}$ and $\lambda \in (0.995,1.005)$ (diamonds).
As the consequence of cross-correlations between $w_i/W$, the volatility correlations for $R(\tau)$ (which is the weight-sum of $w_i/W$ shown in Eq. (7) and equivalent to return $r(\tau)$), defined as the autocorrelation of square of returns, i.e., $x = R^2(\tau)$ in Eq. (8), exhibit the long-range persistence, as shown in Fig. 4 for $\tau = 1$ (lower curve). Fig. 5 gives the results of volatility correlations for $\tau = N = 500$, which exhibits the slow decaying and seems to be of exponential-type (inset of Fig. 5) rather than the power-law decay found in real market data [27]. For correlation of return $R$ the value is around 0, consistent with empirical observations [12,15,17]. This property of volatility clustering can be also reproduced in some microscopic models, but there an extra dynamics like diffusion of traders or feedback between price change and trading activity is needed [28], while in this asynchronous system it is obtained intrinsically.

5. Discussions and conclusions

It is interesting to relate the scales $\tau$ and $N$ here with that of the real markets. The more natural time unit of measurement is $\tau = N$, which corresponds to average updating interval of stocks or agents. To describe the properties of market index, the relevant degrees of freedom $N$ could be interpreted as the major companies capitalization in the market, while for individual stocks $N$ may correspond to the important agents dominating the stock price. Thus, the somewhat different behavior (value of $x$, exponential cut-off, etc.) found in different markets and stocks may be attributed to different effective $N$. Moreover, according to our results in Section 3 the effective $N$ is
not very large (that is, the finite size effect is significant) in real market, indicating a phenomenon that is also found in some microscopic models [29]: it is a limited number of important players, not millions of small traders, who determine the market.

In summary, we have shown that the stochastic multiplicative model (6) reproduces two different power-law behaviors found in reality for the wealth distribution (2) and the return distribution (4), as well as the volatility clustering. The dynamics of returns is described by a walk with steps of sizes obeying a truncated Lévy-like distribution, and in particular, having cross-correlations. These cross-correlations between relative updated wealths \( w_i(t)/W(t) \) are expected to be the origin of the \( \alpha > 2 \) behavior in the tail distribution and the long-range volatility correlations of returns, and can be attributed to the coupling term in multiplicative system (6).

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