PHY8860. Homework 3

This homework assignment is due on September 26. The maximum possible score of this homework, if not turned in by 5 pm that day, will be linearly decreased \( N = N_{\text{max}}(1 - 0.2n) \), where \( n \) is the number of days.

Suggested reading:
M. Peskin and D. Schroeder, “An Introduction to Quantum Field Theory” chapters 6-7.

Problem 1: Computing self-energy.
Consider the theory
\[
\mathcal{L} = \frac{1}{2} \left( \partial \phi \right)^2 - \frac{m^2}{2} \phi^2 - \frac{g}{3!} \phi^3 - \frac{\lambda}{4!} \phi^4. \tag{1}
\]
Find an expression for all one-loop contributions to \( \phi \)'s self-energy and the mass shift \( \delta m \).

Problem 2: Dimensional regularization: computing integrals.
Compute the following integral
\[
I_{\alpha \beta \mu \nu \rho \sigma} = \int d^D k \frac{k_\alpha k_\beta k_\mu k_\nu k_\rho k_\sigma}{(k^2)^n} \tag{2}
\]
in \( D \)-dimensions. For a particular case \( n = 5 \), find the divergent part of this integral in dimensional regularization. **Hint:** First, parameterize the integral. Note that the integral above is completely symmetric with respect to interchange of any of the indices and does not depend on any external momentum.
Problem 3: Pauli-Villars regularization II

Let us learn more about the power of Pauli-Villars regularization, using the simplified version. Consider again a theory of two interacting scalar fields, one of which has a mass $m$ and the other one is massless,

$$L_{\phi\chi} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 + \frac{1}{2}(\partial_\mu \chi)^2 - g\phi^2 \chi .$$  \hspace{1cm} (3)

As you already know, this theory has at least one divergent graph, the self-energy of the $\phi$-field. We have shown that it can be regularized using Pauli-Villars method by doing the substitution

$$\frac{1}{k^2} \rightarrow \frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2}$$  \hspace{1cm} (4)

for the propagator of $\chi$ without touching the propagator of the $\phi$ field.

(i) Show that this substitution can be implemented at the Lagrangian level by introducing a new fictitious field $\xi$ and modifying the Lagrangian in the following way,

$$L = L_{\phi\chi} - L_\xi, \quad L_\xi = \frac{1}{2}(\partial_\mu \xi)^2 - \frac{1}{2}\Lambda^2 \xi^2 - a\phi^2 \xi, \hspace{1cm} (5)$$

where $a$ is some constant. Show that the effect of the new field $\xi$ on the self-energy of the $\phi$-field would be described by Eq. (4) and find the numerical value of $a$.

(ii) Classify all divergencies of the theory described by the Lagrangian $L_{\phi\chi}$. If you decide to completely regularize this theory using Pauli-Villars Lagrangian method, would the modification of the Lagrangian described above in Eq. (5) be sufficient to cure all divergencies? If yes, please motivate your answer. If no, please describe how you would modify the Lagrangian to remove additional divergencies (if any).